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Five- and Eight-Vector Extensions of Relativistic Quantum Theory: The Preferred Reference Frame

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Abstract

The complex quaternion approach to relativistic quantum theory is extended to include time, in the preferred cosmic reference frame, as a fifth dimension. Particle mass appears as an eigenvalue of this invariant operator. Closing the extended algebra also suggests three additional space-time components, possibly related to microscopic rotational degrees of freedom and the scale setting fundamental length. The Dirac, Klein–Gordon, Maxwell, and Einstein equations are generalized to accommodate 5- and 8-vector space-time, including gravitational curvature. The usual conserved probability density is shown to harbor fundamental difficulties.

1. *Introduction*

In a recent series of papers (Edmonds, 1972, 1973), we have attempted to explore the physical possibilities contained in the quaternion formulation of relativistic quantum theory. The basic theme of this work has been the idea that nonrelativistic quantum theory (the Schrödinger equation) is most naturally expressed in the complex number system. Therefore, there may exist a natural hypercomplex number system for relativistic quantum theory. Physicists have chosen instead, to handle the complications of relativistic theory by building matrices on the complex number system. This can apparently always be done, but the resulting structure has many equivalent representations and is so open ended that it does not suggest specific 'natural' generalizations. We have shown that the complex quaternion algebra gives a natural language for space-time and for massless quantum theory. This works only because space has three dimensions. The natural match is unlikely to be a coincidence. Here we develop the idea that rest mass requires a generalization of this algebra and that this mass generalization may be connected with the recently confirmed existence of a preferred reference frame in cosmology.

2. *Massless Quaternion Theory*

To motivate the algebraic extension, we briefly review the structure of massless quantum theory in quaternion form. Four-dimensional space-time is represented by a real quaternion:

$$x \equiv x^\mu e_\mu, \qquad x^0 \equiv ct, \qquad x^k \equiv \{x, y, z\}, \qquad e_0 \leftrightarrow 1, \qquad e_k \leftrightarrow \sigma_k \tag{2.1}$$

A Lorentz transformation takes the form

$$x \to x' \equiv x^{\mu'} e_\mu \equiv \mathscr{L}^* x \mathscr{L}, \qquad \mathscr{L} \equiv \mathscr{L}^\mu e_\mu, \qquad \mathscr{L}\mathscr{L}^\ddagger \equiv e_0 \tag{2.2}$$

We have two basic conjugations, x^* and $x^\ddagger$, with $(xy)^* = y^* x^*$, $(xy)^\ddagger = y^\ddagger x^\ddagger$. For the 4-vector $x^\mu e_\mu$, we have $(x^\mu e_\mu)^* = x^\mu e_\mu$, $(x^\mu e_\mu)^\ddagger = x^0 e_0 - x^k e_k$. It follows that

$$(x|y) \equiv \tfrac{1}{2}[(x^\ddagger y) + (\ )^\ddagger] = x^\mu y_\mu e_0 = \text{invariant} \tag{2.3}$$

Next we define a real quaternion momentum operator, $P^\mu e_\mu$. By definition, it transforms as a 4-vector, $P \to P' = \mathscr{L}^* P \mathscr{L}$. This enables us to write down, by inspection, three covariant wave equations

$$\begin{aligned} &P\psi_a = 0 && \psi_a \to \psi_a' \equiv \mathscr{L}^\ddagger \psi_a \\ &P^\ddagger \psi_v = 0 && \psi_v \to \psi_v' \equiv \mathscr{L}^* \psi_v \\ &(P|P)\psi - P(P|\psi) = 0 && \psi \to \psi' \equiv \mathscr{L}^* \psi \mathscr{L} \end{aligned} \tag{2.4}$$

Notice that no matrices appear here and that the covariance is very manifest.

It is a property of the quaternion algebra that $\{(e_0 + e_3), (e_1 - ie_2)\}$ and $\{(e_1 + ie_2), (e_0 - e_3)\}$ give two independent, two component, quaternion subalgebras, when multiplied only from the left by other quaternions. Thus ψ_a and ψ_v each have only two components.

The wave equations in equation (2.4) have been used to describe neutrinos, antineutrinos, and photons—all massless particles. The complex quaternion algebra seems incapable of accommodating rest mass without resorting to quaternion matrices. This can be taken to mean that a higher algebra is needed in this case, and opens the possibility of a higher dimensional space-time structure.

3. *Extension of the Quaternion Algebra*

The Dirac equation is the only known first-order massive equation of physical importance. We therefore use it to explore the higher algebra. It involves, basically, coupling the ψ_a and ψ_v equations with an invariant parameter, m. There are several covariant ways of doing this. For example

$$\begin{aligned} P\psi_a &= m\psi_v \\ P^\ddagger \psi_v &= m\psi_a \end{aligned} \qquad \text{or alternatively} \qquad \begin{aligned} P^*\psi_a &= m^*\psi_v \\ P^\ddagger \psi_v &= m\psi_a \end{aligned} \tag{3.1}$$

These are equivalent if $P = P^*$ and $m = m^*$. Using matrices, we can write

$$\begin{pmatrix} P & 0 \\ 0 & P^\ddagger \end{pmatrix}\begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & P^\ddagger \\ P & 0 \end{pmatrix}\begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} \tag{3.2}$$

Since the ordinary quaternion elements, e_μ, can also be written in several 2 x 2 matrix representations, σ_μ, we suspect that equation (3.2) is a representation of some higher hypercomplex number algebra. We would like this algebra to have a minimum number of independent elements and to reduce to the quaternion algebra when the rest mass goes to zero. There seems to be a rather unique matrix algebra, of the form in equation (3.2), which meets these two requirements. It gives the Dirac equation in the form

$$\begin{pmatrix} P & 0 \\ 0 & P^{\ddagger *} \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} = \begin{pmatrix} 0 & m^{\ddagger *} \\ m & 0 \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} \tag{3.3}$$

This is equivalent to the usual form because $P^* = P$ and $m^{\ddagger *} = m$. Notice that this matrix algebra is closed under multiplication:

$$\begin{aligned}
\begin{pmatrix} a & 0 \\ 0 & a^{\ddagger *} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{\ddagger *} \end{pmatrix} &= \begin{pmatrix} ab & 0 \\ 0 & (ab)^{\ddagger *} \end{pmatrix}, \\
\begin{pmatrix} 0 & a^{\ddagger *} \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & b^{\ddagger *} \\ b & 0 \end{pmatrix} &= \begin{pmatrix} a^{\ddagger *} b & 0 \\ 0 & (a^{\ddagger *} b)^{\ddagger *} \end{pmatrix}, \\
\begin{pmatrix} a & 0 \\ 0 & a^{\ddagger *} \end{pmatrix} \begin{pmatrix} 0 & b^{\ddagger *} \\ b & 0 \end{pmatrix} &= \begin{pmatrix} 0 & (a^{\ddagger *} b)^{\ddagger *} \\ a^{\ddagger *} b & 0 \end{pmatrix}, \\
\begin{pmatrix} 0 & b^{\ddagger *} \\ b & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{\ddagger *} \end{pmatrix} &= \begin{pmatrix} 0 & (ba)^{\ddagger *} \\ ba & 0 \end{pmatrix}
\end{aligned} \tag{3.4}$$

The matrices $\begin{pmatrix} a & 0 \\ 0 & a^{\ddagger *} \end{pmatrix}$ form a closed subgroup, in one-to-one correspondence with the quaternion multiplication algebra, as desired. The matrices $\begin{pmatrix} 0 & a^{\ddagger *} \\ a & 0 \end{pmatrix}$ give a four-component, complex, extension of the algebra. The new algebra has the following 16 basis elements

$$\begin{pmatrix} e_\mu & 0 \\ 0 & e_\mu{}^{\ddagger} \end{pmatrix}, \quad \begin{pmatrix} ie_\mu & 0 \\ 0 & -ie_\mu{}^{\ddagger} \end{pmatrix}, \quad \begin{pmatrix} 0 & e_\mu{}^{\ddagger} \\ e_\mu & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -ie_\mu{}^{\ddagger} \\ ie_\mu & 0 \end{pmatrix} \tag{3.5}$$

You can try other possibilities in equation (3.2), but they require more than 16 elements to close the algebra.

With this motivation, we can abstract the multiplication properties of equation (3.4) and define the following hypercomplex number system:

$$a = a_e{}^\mu e_\mu + a_f{}^\mu f_\mu, \qquad \mu = 0, 1, 2, 3, \qquad a_e b_e \longleftrightarrow [a_e{}^\mu \sigma_\mu b_e{}^\nu \sigma_\nu]_e$$

$$a_f b_f \longleftrightarrow [(a_f{}^\mu \sigma_\mu)^{\ddagger *} (b_f{}^\nu \sigma_\nu)]_e, \qquad a_f b_e \leftrightarrow [a_f{}^\mu \sigma_\mu b_e{}^\nu \sigma_\nu]_f$$

and

$$a_e b_f \leftrightarrow [(a_e{}^\mu \sigma_\mu)^{\ddagger *} b_f{}^\nu \sigma_\nu]_f \tag{3.6}$$

The $[\ ]_e$ and $[\ ]_f$ mean that the product inside is converted to e_λ and f_λ respectively, after being multiplied out and condensed. For example,

$$(5if_1)(3e_2) \to (5i\sigma_1)(3\sigma_2) = 15i\sigma_1\sigma_2 = -15\sigma_3 \to -15f_3$$

whereas

$$(3e_2)(5if_1) \to (3\sigma_2)^{\ddagger *}(5i\sigma_1) = (-3\sigma_2)(5i\sigma_1) = -15i\sigma_2\sigma_1 = -15\sigma_3 \to -15f_3$$

In particular, this means that

$$(ie_0)(f_0) \to (i\sigma_0)^{\ddagger *}(\sigma_0) = -i\sigma_0 \to -if_0 = -(f_0)(ie_0)$$

As a result, imaginary numbers cannot always be moved through other quantities. This produces considerable complication. One must proceed cautiously in applying the usually valid algebraic processes.

We further define $f_\mu{}^* \equiv f_\mu$ and $\{f_\mu{}^\ddagger\} = \{f_0, -f_k\}$. These are motivated by considering $(e_0f_\mu)^*$ and $(e_0f_\mu)^\ddagger$ and using the multiplication table correspondence for σ_μ. This new number system is much more complicated and algebraically rich than the quaternion algebra.

The new algebra can be thought of as resulting from the generalization

$$a^\mu e_\mu = (a_R{}^\mu + ia_I{}^\mu)e_\mu \to (a^\mu e_0 + b^\mu f_0)e_\mu = a^\mu e_\mu + b^\mu f_\mu \tag{3.7}$$

This means that the wave functions double their degrees of freedom:

$$\psi_a = \psi_+(x)(e_0 + e_3) + \psi_-(e_1 - ie_2) \to (\psi_{e_0+} + \psi_{f_0-})(e_0 + e_3) + (\psi_{e_0-} + \psi_{f_0+})(e_1 - ie_2) \tag{3.8}$$

where ψ_{e_0} is a function of e_0 and ie_0, and ψ_{f_0} is a function of f_0 and if_0. Application of $\frac{1}{2}\hbar e_3$ from the left shows that the ± spin assignments are correct, since $e_3f_0 = f_0(-e_3)$.

4. *Five- and Eight-Vector Wave Equations*

This new algebra immediately suggests an enlargement of our space-time: $x = x^\mu e_\mu \to x_e{}^\mu e_\mu + x_f{}^\mu f_\mu$. Then a Lorentz transformation would be $x \to x' = \mathscr{L}^* x \mathscr{L}$, and we must decide whether $\mathscr{L}_e \to \mathscr{L}_e + \mathscr{L}_f$ or not. Form covariance under Lorentz transformation was originally sold as representing the equivalence of all inertial frames. Einstein then tried to develop a formalism giving the 'equivalence' of all reference frames. In fact, it now appears that there are no physically equivalent reference frames. The recent 3°K background radiation results indicate that there is a preferred reference frame! This leaves Lorentz covariance as a mathematical symmetry of wave equations, rather than a physical statement. Its structure, for a higher dimensional space, is then not obvious. Indeed, we do not know why it should be true at all, even in ordinary space-time! The simplest extension is that where $\mathscr{L} = \mathscr{L}^\mu e_\mu$ and $\mathscr{L}\mathscr{L}^\ddagger = e_0$ are assumed to still hold. Physical laws in the extended space are postulated to be form invariant under the usual Lorentz transformations.

The following equation then naturally occurs:

$$(P_e{}^\mu e_\mu + P_f{}^\mu f_\mu)\psi_a = 0 \qquad \psi_a \to \psi_a' = \mathscr{L}^\ddagger \psi_a \tag{4.1}$$

The Dirac equation is obtained as a special case where

$$P_f{}^0\psi_a = -m\psi_a, \qquad P_f{}^k\psi_a = 0, \qquad \text{and} \qquad (P^\mu e_\mu)^* = P^\mu e_\mu \tag{4.2}$$

We go to the matrix representation

$$\left[\begin{pmatrix} P_e & 0 \\ 0 & P_e{}^{\ddagger *}\end{pmatrix} - \begin{pmatrix} 0 & m^{\ddagger *} \\ m & 0\end{pmatrix}\right]\psi_a = 0 \qquad \psi_a \to \begin{pmatrix}\psi_a \\ \psi_v\end{pmatrix} \tag{4.3}$$

to explicitly display the usual Dirac equation. Having done this once, we forget the matrix representation and explore the structure of equation (4.1) with regard to covariance and the Klein–Gordon operator.

By assumption, $P = P^*$ and $P' = \mathscr{L}^* P \mathscr{L}$ and $\mathscr{L} = \mathscr{L}_e$. As a consequence

$$\mathscr{L}^*(P_f{}^0 f_0)\mathscr{L} = (P_f{}^0 f_0)\mathscr{L}^\ddagger \mathscr{L} = P_f{}^0 f_0 \tag{4.4}$$

Thus $P_f{}^0$ is invariant if $P_f{}^k = 0$. (We shall see that it is invariant even if $P_f{}^k \neq 0$.) The usual quantum postulate $P_e{}^\mu \leftrightarrow \partial_e{}^\mu$, suggests that $P_f{}^0$ is associated with $\partial_f{}^0$, i.e. with $\partial/\partial T$ where T is an invariant time. The obvious 'basic time' for the universe is the time since the 'big bang' in the preferred reference frame—'cosmic time'. The Dirac equation shows that $P_f{}^0 \leftrightarrow m$, the present rest mass of a particle. We thus obtain a connection between 'eigenstates' of cosmic time and rest mass. The universe is apparently expanding as cosmic time moves forward. The conjectured connection between rest mass and eigenstates of the cosmic time operator may also indicate a connection between rest energy and the age of the universe. If rest energies are increasing, would we not see old photons, from distant atoms, redshifted (the 'tired light idea')? The slowly increasing energy of the universe would also affect the expansion rate. This speculation about the time dependence of rest mass is not a necessary consequence of the connection between mass and cosmic time. It does pose some interesting possibilities. The recent indications of possible failure of time reversal symmetry may also be tied in with the appearance of absolute time in the extended quaternion algebra and the expansion of the universe.

An 8-vector has the form $P_e{}^\mu e_\mu + P_f{}^\mu f_\mu$. A corresponding 8-vector event would be characterized by $(t, x^k, T, \alpha, \beta, \gamma)$. Our macroscopic experience would indicate that α, β, and γ are not macrocoordinates. They could, however, be subatomic degrees of freedom. If they are cyclic, e.g. $\alpha + l_0 = \alpha$, they could be used to introduce a fundamental length into physics. This, along with c and $\hbar$, would set the scale of the universe. But what would these coordinates mean physically?

Let us look at the classical definition of 4-momentum to suggest an answer. We have $(dt, dx^k, dT, d\alpha^k, m)$ parametrising the movement of a mass, m. From dt and dx^k in e_μ, we have the invariant $d\tau = (dt^2 - (dx^k)^2)^{1/2}$. From dT and $d\alpha^k$ in f_μ, we have the invariant dT. Therefore the 'symmetric' way to define the classical momentum 8-vector is

$$P \equiv P_e{}^{\mu} e_{\mu} + P_f{}^{\mu} f_{\mu} \equiv m \frac{dt}{d\tau} e_0 + m \frac{dx^k}{d\tau} e_k + m \frac{dT}{dT} f_0 + m \frac{d\alpha^k}{dT} f_k \quad (4.5)$$

This gives the usual form when $d\alpha^k/dT$ is negligible. It seems quite 'natural' to identify $m(d\alpha^k/dT)$ with the spin degrees of freedom of a classical body. An event in space-time, characterising the instantaneous state of a classical particle, would require specifying its mass, local frame time and particle position, cosmic time, and the direction and magnitude of its angular velocity. Hence, the very peculiar α^k coordinates and momenta $P_f{}^k$, may be related, quantum mechanically, to intrinsic angular momentum (spin or tumble) for the fundamental particle. As we shall see, these components transform among themselves under Lorentz transformation, with $P_f{}^k P_f{}^k$ frame invariant. Under space rotations they transform similarly to $P_e{}^k$. Without the existence of a preferred frame, they would be more suspect, since the $P_f{}^k$ are all real, classically, *only* in one Lorentz frame! One cannot help being reminded of the complex angular momentum ideas of Regge pole theory.

Despite these tantalizing possibilities, the inclusion of $P_f{}^k$ momenta operators, produce sweeping complications in the wave equation. Certainly the world would be much simpler to deal with, if these components are zero.

To illustrate the subtleties of this algebra we consider the simplest problem —a free particle at rest. At rest is taken to mean that $P_e{}^k\psi = P_f{}^k\psi = 0$. We then have

$$(P_e{}^0 e_0 + P_f{}^0 f_0)\psi = 0 \quad (4.6)$$

To go further, we must now postulate the operator nature of P^{μ}. From our past experience, we would be inclined to try $P^{\mu}_{e,f} \to i\hbar\partial^{\mu}_{e,f}$. This actually creates real problems. Recall that we have postulated that $P = P^*$. Then we would need $(i\hbar\partial^{\mu}_{e,f})^* = (i\hbar\partial^{\mu}_{e,f})$. Since $(i\hbar)^* = -i\hbar$, we run up against $\partial^{\mu *} = -\partial^{\mu}$. In the old Dirac theory, we interpret $(i\hbar\partial)^*$ as an Hermitian conjugate with respect to an inner product. Here, no such inner product has been defined. This problem seems to be very important. It is reflected in the problem of finding a conserved current, as we shall see later.

Recall that the nonrelativistic limit is formed from the Klein–Gordon equation. In this equation $i\hbar\partial^{\mu}$ does not appear directly; we have

$$[(i\hbar\partial^{\mu})(i\hbar\partial_{\mu}) - m^2]e_0\psi = [-\hbar^2\partial^{\mu}\partial_{\mu} - m^2]e_0\psi = 0 \quad (4.7)$$

Trying to factor this, is where Dirac originally started. To get the nonrelativistic limit, we rewrite this as

$$\begin{aligned}[-\hbar^2\partial^0\partial^0 - m^2]e_0\psi &= [+\hbar^2\partial^k\partial_k]\psi \\ &= [i\hbar\partial^0 - m][i\hbar\partial^0 + m]\psi \approx [i\hbar\partial^0][2m]\psi \quad (4.8)\end{aligned}$$

Thus, any first-order equation which gives equation (4.7), will automatically become the Schrödinger equation in the nonrelativistic limit.

We can use the multiplication properties of the quaternions to rewrite equation (4.7) in the form

$$[(i\hbar\partial^{\mu}e_{\mu})^{\ddagger}(i\hbar\partial^{\nu}e_{\nu}) - m^2 e_0]\psi = 0 \tag{4.9}$$

Then by postulating $i\hbar\partial^{\mu}e_{\mu} \to i\hbar\partial^{\mu}e_{\mu} - \epsilon A^{\mu}e_{\mu}$, we obtain the Pauli equation in the nonrelativistic limit. This approach, however, doesn't treat m^2 as the 'square' of a momentum quaternion element. Also, when A is introduced we have to write

$$[\tfrac{1}{2}[\{(i\hbar\partial^{\mu}e_{\mu} - \epsilon A^{\mu}e_{\mu})^{\ddagger}(i\hbar\partial^{\nu}e_{\nu} - \epsilon A^{\nu}e_{\nu})\} + \{\ \}^{\ddagger}] - m^2 e_0]\psi = 0 \tag{4.10}$$

because ∂^{μ} and A^{ν} do not commute. We only want the e_0 term, so $\{\ \}^{\ddagger}$ is added to cancel off the e_k terms.

To include m on an equal footing in the factorisation we go back to the Klein-Gordon equation with A^{μ} interaction

$$[(i\hbar\partial^{\mu} - \epsilon A^{\mu})(i\hbar\partial_{\mu} - \epsilon A_{\mu}) - m^2]\psi = 0 \tag{4.11}$$

This is known to be an accurate equation for charged, spin zero particles. Multiplying through by $(-i)^2 = -1$ we get

$$[(\hbar\partial^{\mu} + i\epsilon A^{\mu})(\hbar\partial_{\mu} + i\epsilon A_{\mu}) + m^2]\psi = 0 \tag{4.12}$$

This form suggests the extended quaternion factorisation

$$[P_e + P_f]\psi = 0, \qquad P_e{}^{\mu} \to \hbar\partial^{\mu} + i\epsilon A^{\mu}, \qquad P_f{}^{\mu} \to \{m, 0\} \tag{4.13}$$

Then

$$\begin{aligned}\{(P_e{}^{\ddagger} + P_f{}^{*})(P_e + P_f)\} + \{\ \}^{\ddagger} &= \{P_e{}^{\ddagger}P_e\} + \{\ \}^{\ddagger} + \{mf_0P_e + mP_e{}^{\ddagger}f_0\} + \{\ \}^{\ddagger} \\ \{P_f{}^{*}P_f\} + \{\ \}^{\ddagger} &= 2[(\hbar\partial^{\mu} + i\epsilon A^{\mu})(\hbar\partial_{\mu} + i\epsilon A_{\mu}) + m^2]e_0 + 2[m\hbar\partial_e{}^0]f_0\end{aligned} \tag{4.14}$$

We see that equation (4.13) gives the e_0 term in the Klein-Gordon equation. Therefore, we could take as our basic postulate $P_e{}^{\mu} \to \hbar\partial^{\mu}$, with the electromagnetic coupling $+i\epsilon A^{\mu}$. Notice that $(\hbar\partial^{\mu})^* = \hbar\partial^{\mu}$ but $(i\epsilon A^{\mu})^* = -(i\epsilon A^{\mu})$, assuming A^{μ} is real as usual.

The free particle rest state equation becomes

$$[e_0\hbar\partial_e{}^0 + mf_0]\psi = 0 \tag{4.15}$$

The solution is

$$\psi = \left[e_0 \cosh\left(\frac{mt}{\hbar}\right) - f_0 \sinh\left(\frac{mt}{\hbar}\right)\right][C_1(e_0 + e_3) + C_2(e_1 - ie_2)] \tag{4.16}$$

where C_1 and C_2 are arbitrary complex numbers. The solution is easy to verify because ∂^0 commutes with f_0. We have used $e_0 = f_0f_0$ also.

Since this solution is not oscillatory, it is probably useless. If we go back to equation (4.11), we can try the factorisation

$$[P_e + P_f]\psi = 0, \qquad P_e{}^{\mu} \to i\hbar\partial^{\mu} - \epsilon A^{\mu}, \qquad P_f{}^{\mu} \to \{-m, 0\} \tag{4.17}$$

Then

$$\{(P_e^{\ddagger} - P_f^*)(P_e + P_f)\} + \{\ \}^{\ddagger} = \{P_e^{\ddagger}P_e\} + \{\ \}^{\ddagger} - \{m^2 e_0\} - \{\ \}^{\ddagger}$$
$$+ \{-mf_0 P_e + mP_e^{\ddagger} f_0\} + \{\ \}^{\ddagger}$$
$$= 2[(i\hbar\partial^\mu - \epsilon A^\mu)(i\hbar\partial_\mu - \epsilon A_\mu) - m^2]e_0$$
$$+ [-m(i\hbar\partial^0 - \epsilon A^0) + m(i\hbar\partial^0 - \epsilon A^0)^*]f_0 \tag{4.18}$$

Therefore, we again obtain the e_0 term in the Klein-Gordon equation. In this case our basic postulate is $P_e^\mu \to i\hbar\partial^\mu \equiv (i\hbar\partial^\mu)^*$, with the electromagnetic coupling $-\epsilon A^\mu \equiv (-\epsilon A^\mu)^*$. Notice that this causes the f_0 term in equation (4.18) to cancel.

The free particle rest state equation now becomes

$$[e_0 i\hbar\partial^0 - mf_0]\psi = 0 \tag{4.19}$$

The solution is more subtle and relies on $(i\hbar\partial^0) \equiv (i\hbar\partial^0)^*$:

$$\psi_{\text{rest}} = \left[\left[e_0 \cos\frac{mt}{\hbar}\right] - \left[f_0 i \sin\frac{mt}{\hbar}\right]\right] [C_1(e_0 + e_3) + C_2(e_1 - ie_2)] \tag{4.20}$$

The critical step is

$$[e_0 i\hbar\partial^0]\left[f_0 i \sin\frac{mt}{\hbar}\right] = \left[f_0(i\hbar\partial^0)^*\left(i \sin\frac{mt}{\hbar}\right)\right] = \left[f_0\left(-m \cos\frac{mt}{\hbar}\right)\right] \tag{4.21}$$

and use is made of $-ie_0 = f_0(-if_0)$ to factor mf_0 out to the left. If we consider $(ie_0)\psi_{\text{rest}}$, we find that $(ie_0)(mf_0)\psi_{\text{rest}} = -(mf_0)(ie_0)\psi_{\text{rest}}$, which means that $(ie_0)\psi_{\text{rest}}$ is the solution to equation (4.19) if $m \to -m$. This appears to correspond to cosmic time reversal, $T \to -T$. (The expanding universe then suggests a natural asymmetry between matter and antimatter.) For a given sign of m we do *not* appear to obtain 'antiparticle' solutions! Notice also that $\exp i\theta\,\psi_{\text{rest}}$ is not a solution unless $\theta = 0$, because mf_0 will not commute with the imaginary part of $\exp i\theta$. The old arbitrary phase factor is also gone! Any constant quaternion, multiplying ψ_{rest} from the right, leaves it a solution.

We can now consider

$$-mf_0\psi \equiv [f_0 ik\partial_f^0]\psi \equiv f_0[e_0 ik\partial_f^0]\psi \tag{4.22}$$

Is this compatible with equation (4.19) and the solution (4.20)? It is if, again we define $(ik\partial_f^0)^* = ik\partial_f^0$, where k is an undefined real constant (possibly cosmological). We can then carry $(e_0 ik\partial_f^0)$ through $[f_0 i \sin(mt/\hbar)]$ and obtain

$$\psi_{\text{rest}} = \left(\left[e_0 \cos\frac{mt}{\hbar}\right] - \left[f_0 i \sin\frac{mt}{\hbar}\right]\right) \exp\left(\frac{-mT}{ik}\right)$$
$$\times [C_1(e_0 + e_3) + C_2(e_1 - ie_2)] \tag{4.23}$$

Notice that m can be pulled through the $f_0 \sin(mt/\hbar)$ term only if it is real. There appear to be no spin $\frac{1}{2}$ tachyons in this algebra!

Under a Lorentz transformation, we obtain 'plane wave states'

$$\psi_{a(\text{rest})} \to \psi_{a(\text{planewave})} = \mathscr{L}^{\ddagger}\psi_{a(\text{rest})} \tag{4.24}$$

For example, a boost along x^1 corresponds to $\mathscr{L} = \cosh(\theta/2)e_0 + \sinh(\theta/2)e_1$. The equation (4.18) suggests the covariant equation

$$(P_e{}^{\ddagger} - P_f{}^{*})\psi_v = 0 \qquad \psi_v \to \psi_v' = \mathscr{L}^{*}\psi_v \tag{4.25}$$

It gives the cosmic time reversed *rest* solutions of equation (4.19) and $\psi_{v(\text{rest})} = \pm i\psi_{a(\text{rest})}$. Later we can show that it is covariant. These wave equations for ψ_a and ψ_v are the 8-vector analogues of the massless Weyl wave equations for ψ_a and ψ_v.

We seem to have achieved a decoupling of positive and negative 'energy' states, at the cost of having to deal with functions of a noncommutative ring algebra $C_1e_0 + C_2f_0$, instead of the field algebra C_1e_0. (The numbers f_0 and if_0 are solutions to the problem $x^2 = 1$ and $ix + xi = 0$.) This is a high price to pay, but the possibility that the absolute time operator may illuminate the particle spectrum, since it acts as a mass eigenstate operator, may make the added complications worth the effort. The eight component set $\{e_\mu, f_\mu\}$ also reminds one of the '8 fold way'. We can form groups with it similar to $SU(3)$ except that we would expect 8 basic partons, with antiparticles, instead of three quarks. After all, this algebra was produced by the existence of rest mass in physics.

Neither should the $P_f{}^k$ terms be dismissed lightly. It has been known for some time that rotations of 360° do not give $\mathscr{L} = e_0$, but rather $\mathscr{L} = ie_3$ for rotations around the z-axis. Thus ψ is not invariant. Notice that

$$\mathscr{L}^{*}f_1\mathscr{L} = (ie_3)^{*}f_1(ie_3) = f_2(ie_3) = -f_1 \tag{4.26}$$

so that $\{P_f{}^k f_k\}$ changes some signs under a 360° rotation of our coordinates. It then does not correspond to ordinary angular momentum. It may someday explain the reason why two complete rotations are required to bring you back to the same orientation with respect to the universe.

5. *Eight-Vector Inner Products: Maxwell's Equation*

In order to write down a generalised Maxwell equation, we need to define the inner product $(P|A)$, as well as $(P|P)$, in a covariant manner. We have seen, in equations (4.14) and (4.18), that two conjugations can be considered in addition to $*$ and $\ddagger$, in forming the generalised inner product:

$$(P_e + P_f)^{\wedge} \equiv P_e{}^{\ddagger} + P_f{}^{*}, \quad (P_e + P_f)^{\vee} \equiv (P_e{}^{\ddagger} - P_f{}^{*}), \qquad (P_e + P_f)^{\wedge\vee} = (P_e - P_f) \tag{5.1}$$

We know that in the usual complex quaternion (massless) case

$$(P_e|A_e) \equiv \tfrac{1}{2}[(P_e{}^{\ddagger}A_e) + (\ )^{\ddagger}] = P_e{}^{\mu}A_{e\mu}e_0 \tag{5.2}$$

This needs to be generalised for 8-vectors. In order to examine possible generalisations, we need to investigate composite conjugations, which follow from the

e and f multiplication table. This is straightforward but tedious. The following results are readily verified:

$$(e_a e_b)^* = e_b{}^* e_a{}^*, \qquad (e_a e_b)^\ddagger = e_b{}^\ddagger e_a{}^\ddagger, \qquad (f_a f_b)^* = f_b{}^\ddagger f_a{}^\ddagger,$$
$$(f_a f_b)^\ddagger = f_b{}^* f_a{}^*, \qquad (ef)^* = f^* e^\ddagger, \qquad (fe)^* = (e^\ddagger f^*), \qquad (ef)^\ddagger = f^\ddagger e^*,$$
$$(fe)^\ddagger = e^* f^\ddagger$$
$$(ab)^{\ddagger *} = a^{\ddagger *} b^{\ddagger *}, \qquad (ab)^\wedge = b^\wedge a^\wedge \qquad (ab)^\vee = b^\vee a^\vee$$
$$(e_a f e_b)^* = e_b{}^\ddagger f^* e_a{}^\ddagger, \qquad (e_a f e_b)^\ddagger = e_b{}^* f^\ddagger e_a{}^*, \qquad (f_a e f_b)^* = f_b{}^\ddagger e^* f_a{}^\ddagger,$$
$$(e f_a f_b)^* = f_b{}^\ddagger f_a{}^\ddagger e^*, \qquad (f e_a e_b)^* = e_b{}^\ddagger e_a{}^\ddagger f^*, \qquad (f_a f_b f_c)^* = f_c{}^* f_b{}^* f_a{}^* \tag{5.3}$$

We can now calculate

$$(\mathscr{L}^* P \mathscr{L})^\vee (\mathscr{L}^* A \mathscr{L}) = [(\mathscr{L}^* P_e \mathscr{L})^\ddagger - (\mathscr{L}^* P_f \mathscr{L})^*] \mathscr{L}^* A \mathscr{L}$$
$$= \mathscr{L}^\ddagger [P^\vee (\mathscr{L}^{\ddagger *} \mathscr{L}^*) A] \mathscr{L}$$
$$(\mathscr{L}^* P \mathscr{L})(\mathscr{L}^* A \mathscr{L})^\vee = \mathscr{L}^* P \mathscr{L} [(\mathscr{L}^* A_e \mathscr{L})^\ddagger - (\mathscr{L}^* A_f \mathscr{L})^*]$$
$$= \mathscr{L}^* [P(\mathscr{L} \mathscr{L}^\ddagger) A^\vee] \mathscr{L}^{\ddagger *} \tag{5.4}$$

This shows that these terms transform differently, therefore they cannot be added to form an innerproduct. Notice, however, that

$$(P^\vee A)^\wedge \to (\mathscr{L}^\ddagger P^\vee A \mathscr{L})^\wedge = (\mathscr{L}^\ddagger (e + f) \mathscr{L})^\wedge = (\mathscr{L}^\ddagger e \mathscr{L})^\wedge + (\mathscr{L}^\ddagger f \mathscr{L})^\wedge$$
$$= \mathscr{L}^\ddagger e^\ddagger \mathscr{L}^{\ddagger\ddagger} + \mathscr{L}^\ddagger f^* \mathscr{L}^{\ddagger\ddagger} = \mathscr{L}^\ddagger (e + f)^\wedge \mathscr{L} \tag{5.5}$$

So we can form an inner product given by

$$P^\vee A + (P^\vee A)^\wedge \to [\mathscr{L}^\ddagger (e + f) \mathscr{L}] + [\ \ ]^\wedge = \mathscr{L}^\ddagger [e + e^\ddagger + f + f^*] \mathscr{L}$$
$$= \mathscr{L}^\ddagger [e + e^\ddagger] \mathscr{L} + \mathscr{L}^\ddagger [f + f^*] \mathscr{L} = e + e^\ddagger + \mathscr{L}^\ddagger [f + f^*] \mathscr{L} \tag{5.6}$$

This inner product is not invariant. It transforms like $\mathscr{L}^\ddagger(\ )\mathscr{L}$ and has an invariant part, $e + e^\ddagger$. In the limit P_f and $A_f \to 0$, this gives the old result for the massless 4-vector quaternion inner product, $P_e{}^\mu A_{e\mu} e_0$. For the special 8-vector case $A \to P$, we get

$$P^\vee P + (P^\vee P)^\wedge = (P_e{}^\mu P_{e\mu} - P_f{}^\mu P_{f\mu}) e_0 + [(P_e{}^\ddagger P_f) - (P_e{}^\ddagger P_f)^*] + [\ \ ]^* \tag{5.7}$$

Since $[f - f^*] + [\ \ ]^* = 0$, the noninvariant part of this inner product is zero, for the norm of an 8-vector.

These two results for $(P|P)$ and $(P|A)$ are exactly what are needed to write the covariant 8-vector Maxwell equation!

$$(P|P)A - P(P|A) = J, \qquad A \to A' = \mathscr{L}^* A \mathscr{L}, \qquad J \to J' = \mathscr{L}^* J \mathscr{L},$$
$$(P|P) \to (P'|P') = (P|P) \propto e_0, \qquad P \to P' = \mathscr{L}^* P \mathscr{L}, \qquad (P|A) \to (P'|A') = \mathscr{L}^\ddagger (P|A) \mathscr{L},$$
$$(P|A) \equiv \tfrac{1}{2}[(P^\vee A) + (P^\vee A)^\wedge], \qquad P'(P'|A') = \mathscr{L}^* P(P|A) \mathscr{L} \tag{5.8}$$

Notice that

$$J \equiv J^* = (J_e + J_f)^* \to (\mathscr{L}^* J \mathscr{L})^* = (\mathscr{L}^* J_e \mathscr{L})^* + (\mathscr{L}^* J_f \mathscr{L})^*$$
$$= \mathscr{L}^* J_e{}^* \mathscr{L} + \mathscr{L}^\ddagger J_f{}^* \mathscr{L}^{\ddagger *} \neq \mathscr{L}^* (J_e + J_f) \mathscr{L} = J' \tag{5.9}$$

Again we see the subtle complications of 8-vectors. In general, if J is more than a 5-vector, it does not produce a real photon field A, since $J = J^*$ is not a Lorentz invariant concept for $J_f^k \neq 0$. Zero mass photons are characterised by $P_f^0 A = 0$, but this does not imply that $P_f^k A = 0$, so that A could still in general be an 8-vector.

This discussion leads us to the problem of finding an 8-vector current vector for the generalised Dirac equation. We would like it to be real and conserved, if possible (at least the J_e part). We suspect, already, that the P_f^k operators will cause one or both of these requirements to fail. We would also suspect that $\psi^\wedge$ rather than $\psi^\vee$ (which introduces a minus sign) will play a role, since the j^0 component 'should' be real in the limit $P_f^k \to 0$.

6. *Eight-Vector Dirac Current*

The question of Lorentz covariance is most complicated for the Dirac current problem. This was true in the old quaternion matrix formalism we used previously. There we had

$$\begin{pmatrix} P & 0 \\ 0 & P^\ddagger \end{pmatrix}\begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix} \quad \text{and} \quad \left\{\left[(\psi_a{}^*\psi_v{}^*)\begin{pmatrix} e_\mu & 0 \\ 0 & e_\mu{}^\ddagger \end{pmatrix}\begin{pmatrix} \psi_a \\ \psi_v \end{pmatrix}\right] + [\ ]^\ddagger\right\} e_\mu = j^\mu e_\mu \tag{6.1}$$

Here, for the first time, quaternion basis elements (e_μ and $e_\mu{}^\ddagger$) appear directly in the simplest expression of the desired quaternion 4-vector current. We must decide how they are to transform under a Lorentz transformation. This was answered by considering the more general case of quantum theory in curved space-time. There it was found that $e_\mu \to b^\mu(x)$ in j^μ, so that $J = j^\mu b_\mu(x)$ is invariant for any general coordinate change. The Lorentz principle thus involves $\psi_a \to \mathscr{L}^\ddagger \psi_a$, $\psi_v \to \mathscr{L}^* \psi_v$, $b^\mu(x) \to \mathscr{L}^* b^\mu \mathscr{L}$, and $b_\mu(x) \to \mathscr{L}^* b_\mu \mathscr{L}$, such that the equation containing $j^\mu b_\mu$ is form invariant. We find then

$$\begin{aligned} \psi_a{}^* e_\mu \psi_a &\to \psi_a'^* e_\mu' \psi_a' = (\mathscr{L}^\ddagger \psi_a)^* (\mathscr{L}^* e_\mu \mathscr{L})(\mathscr{L}^\ddagger \psi_a) = \text{invariant} \\ \psi_v{}^* e_\mu{}^\ddagger \psi_v &\to \psi_v'^* e_\mu'^\ddagger \psi_v' = (\mathscr{L}^* \psi_v)^* (\mathscr{L}^* e_\mu \mathscr{L})^\ddagger (\mathscr{L}^* \psi_v) = \text{invariant} \end{aligned} \tag{6.2}$$

The $[\ ]^\ddagger$ term in j^μ is included so that $\mathscr{L}^*$ can be pulled through j^μ to the left. Then $j^\mu e_\mu \to j^{\mu\prime} e_\mu' = j^\mu e_\mu' = j^\mu \mathscr{L}^* e_\mu \mathscr{L} = \mathscr{L}^* j^\mu e_\mu \mathscr{L}$, under a Lorentz transformation. A similar kind of approach should work in the extended quaternion algebra.

We first look for an invariant of the general form

$$\psi_a{}^? g \psi_a \qquad \text{where } g = e \text{ or } f \tag{6.3}$$

Under a Lorentz transformation, this becomes

$$(\mathscr{L}^\ddagger \psi_a)^? \mathscr{L}^* g \mathscr{L} \mathscr{L}^\ddagger \psi_a \Rightarrow (\mathscr{L}^\ddagger \psi_a)^? = \psi_a{}^? \mathscr{L}^{\ddagger *} \tag{6.4}$$

The conjugation with the general property $(ab)^? = b^? a^?$ is $(ab)^\wedge$. However, $\mathscr{L}^{\ddagger\wedge} = \mathscr{L}^{\ddagger\ddagger} = \mathscr{L}$. Since $(ab)^{\ddagger *} = a^{\ddagger *} b^{\ddagger *}$ in general for the extended algebra, we are saved. Let $? = \wedge\ddagger *$. Then $\mathscr{L}^{\ddagger\wedge\ddagger *} = \mathscr{L}^{\ddagger\ddagger\ddagger *} = \mathscr{L}^{\ddagger *}$ as desired. Notice that all this works out only if $\mathscr{L} = \mathscr{L}_e$. Earlier, we assumed this would be true for the Lorentz principle in the extended algebra. Here we see that it is a very important restriction. Our expressions for j^μ will look very similar to those in second quantised conventional quantum theory, if $\wedge\ddagger * \equiv \dagger$. In the second quantised context, † refers to the Fock space Hermitian conjugate. Our present considerations are restricted to an unquantised approach, so † is just an abbreviation for $\wedge\ddagger *$. Therefore,

$$(ab)^\dagger = b^\dagger a^\dagger, \qquad e^\dagger = e^{\wedge\ddagger *} = e^*, \qquad f^\dagger = f^{\wedge\ddagger *} = f^\ddagger \tag{6.5}$$

We now have

$$j^\mu \propto \psi_a{}^\dagger g_\mu \psi_a = (\psi_a{}^\dagger g_\mu{}^\dagger \psi_a)^\dagger, \qquad e_\mu{}^\dagger = e_\mu, \qquad f_\mu{}^\dagger = f_\mu{}^\ddagger \tag{6.6}$$

so j^μ is a Lorentz invariant, and we must add something to it so that it becomes $\propto e_0$. In general

$$a \equiv e + f \equiv a^\dagger = e^* + f^\ddagger \Rightarrow a_e{}^\mu = a_e^{\mu *}, \qquad a_f{}^k = 0 \tag{6.7}$$

Therefore

$$(\psi_a{}^\dagger e_\mu \psi_a)^\dagger = (\psi_a{}^\dagger e_\mu \psi_a) \Rightarrow (\psi_a{}^\dagger e_\mu \psi_a) \equiv a_e{}^\mu e_\mu + a_f{}^\mu f_\mu \tag{6.8}$$

has $a_e{}^\mu = a_e^{\mu *}$ and $a_f{}^k = 0$, but $a_f{}^0$ unspecified. The same general properties would apply for $\psi_a{}^\dagger f_0 \psi_a$. Note that

$$(\psi_a{}^\dagger f_k \psi_a)^\dagger = -(\psi_a{}^\dagger f_k \psi_a) \Rightarrow (\psi_a{}^\dagger f_k \psi_a) \equiv a_e{}^\mu e_\mu + a_f{}^\mu f_\mu \tag{6.9}$$

has $a_e{}^\mu = -a_e{}^{\mu *}$ and $a_f{}^0 = 0$, but $a_f{}^k$ unspecified. Once again, f_k provides complications. These f_k terms are the only g_μ elements in $\psi^\dagger g_\mu \psi$ which do not have a real e_0 component.

According to the invariance principle, we cannot just write

$$J = [\psi^\dagger g_\mu \psi |_{e_0 \,\text{component}}] g_\mu \tag{6.10}$$

We must, instead, add enough conjugation terms to $\psi^\dagger g_\mu \psi$ to cancel the e_k, f_0, and f_k terms. This seems to automatically eliminate the e_0 contribution of e_μ and f_0 or of f_k in j^μ. We supposedly must keep the e_μ contributions, $\psi^\dagger e_\mu \psi$, therefore it seems that $J = j^\mu g_\mu$ is only a 5-vector and is real: $j_f{}^k = 0, j_e^{\mu *} = j_e{}^\mu$, $j_f{}^0 = j_f{}^{0 *}$.

Explicitly,

$$J = \tfrac{1}{4}\{[(\psi^\dagger g_\mu \psi) + (\quad)^{\vee\ddagger *}] + [\quad]^{\vee}\} g_\mu = j_e{}^\mu e_\mu + j_f{}^0 f_0 \tag{6.11}$$

In curved space $g_\mu \to g_\mu(x)$. (Notice then that $\psi^\dagger e_0 \psi = \psi^\dagger \psi$ in flat space but $\psi^\dagger b_0(x) \psi \neq \psi^\dagger \psi$ in curved space!) We can display $j_e{}^0$ directly, using

$$\begin{aligned}
\psi_a &\equiv F_1[e_0, f_0](e_0 + e_3) + F_2[e_0, f_0](e_1 - ie_2) \\
&= G_{1+}[e_0](e_0 + e_3) + G_{1-}[e_0] f_0 (e_0 + e_3) + G_{2+}[e_0] f_0 (e_1 - ie_2) \\
&\quad + G_{2-}[e_0](e_1 - ie_2) \\
&= [G_{1+}(e_0 + e_3) + G_{2-}(e_1 - ie_2)] + [G_{2+}(f_1 - if_2) + G_{1-}(f_0 + f_3)] \\
&\equiv [\psi_e] + [\psi_f]
\end{aligned} \tag{6.12}$$

We find that

$$j^0 = (\psi^\dagger e_0 \psi)|_{e_0\,\text{part}} = (\psi_e + \psi_f)^\dagger(\psi_e + \psi_f)|_{e_0} = (\psi_e{}^* + \psi_f{}^\ddagger)(\psi_e + \psi_f)|_{e_0}$$
$$= |G_{1+}|^2 + |G_{1-}|^2 + |G_{2+}|^2 + |G_{2-}|^2 \tag{6.13}$$

Notice that functions $F[e_0, f_0]$ and $G[e_0]$ in ψ have the properties

$$F[e_0, f_0]^\dagger F[e_0, f_0] = (G_+ e_0 + G_- f_0)^\dagger(G_+ e_0 + G_- f_0)$$
$$= (G_+{}^* e_0 + G_- f_0)(G_+ e_0 + G_- f_0) = (|G_+|^2 + |G_-|^2)e_0$$
$$+ (2G_+ G_-)f_0 \tag{6.14}$$

If we use the notation $\vee\ddagger{*} \equiv \circledast$, then $(ae_0 + bf_0)^\circledast = a^*e_0 - bf_0$. We can now write equation (6.13) in the form

$$j^0 = \tfrac{1}{2}[(F_1{}^\dagger F_1) + (\ )^\circledast] + \tfrac{1}{2}[(F_2{}^\dagger F_2) + (\ )^\circledast]$$
$$F_1{}^\circledast\ F_1 = (|G_+|^2 - |G_-|^2)e_0 + Of_0 \tag{6.15}$$

This provides a positive definite norm for the number system $C_1e_0 + C_2f_0$. We further note that

$$(C_1e_0 + C_2f_0)(C_3e_0 + C_4f_0) = (C_1C_3 + C_2{}^*C_4)e_0 + (C_2C_3 + C_1{}^*C_4)f_0 \equiv 0$$
$$\Rightarrow C_1C_3 + C_2{}^*C_4 = 0 \qquad \text{and} \qquad C_2C_3 + C_1{}^*C_4 = 0$$

This can be satisfied for $C_3 = C_4$ and $C_1 = -C_2{}^*$. Therefore, the number system $\{C_1e_0 + C_2f_0\}$ has zero divisors in it and, as the second equation in (6.15) shows, the cancellation law holds only when $|C_1|^2 \neq |C_2|^2$. This is definitely a ring, rather than a field.

We concluded above that J should be a 5-vector, instead of an 8-vector. Under a Lorentz transformation this gives

$$J' = \mathscr{L}^*(J_e + j_f{}^0 f_0)\mathscr{L} = \mathscr{L}^* J_e \mathscr{L} + j_f{}^0 f_0 \mathscr{L}^\ddagger \mathscr{L} \tag{6.16}$$

which shows that $j_f{}^0$ is a scalar field. Explicitly, it is

$$j_f{}^0 = \psi^\dagger f_0 \psi|_{e_0\,\text{part}} = (\psi_e + \psi_f)^\dagger f_0(\psi_e + \psi_f)|_{e_0}$$
$$= [(G_{1+}^* G_{1-} + G_{2+}^* G_{2-}) + (\ )^*]e_0 \tag{6.17}$$

Referring back to equation (4.20) for ψ_{rest}, we see that $j_f{}^0 = 0$ in this case. This must also be true then for $j_f{}^0$ representing a 'plane wave' particle state (constant **V**). It seems likely that A has an $A_f{}^0$ scalar component if $j_f{}^0$ is ever nonzero. This would represent a scalar electromagnetic field component.

We see that J, as a source of A, has three distinct parts: J_e, $j_f{}^0 f_0$, and $j_f{}^k f_k$. To couple these to A, we must introduce scalar charges ($\propto e_0$) in front of each type. It would be presumptuous to expect that all these charges are equal. In fact, some may be zero, in order to satisfy some other general feature besides Lorentz covariance, e.g. a conservation law. Also, there could be other A type fields described by the Maxwell form but with their own sources, e.g. $j_f{}^k f_k$ alone, as a source.

The same can be said of the $A\psi_a$ couplings, since A in general can have three distinct parts: $A_e, A_f{}^0 f_0$, and $A_f{}^k f_k$. These parts are each self-contained under Lorentz transformation. The unique, 'one-sided' transformation structure of ψ_a and ψ_v, allows us to also consider $A\psi_a\epsilon$ couplings, where now $\epsilon = \epsilon_e + \epsilon_f$, in general, and is invariant. The coupling $\epsilon_f{}^0 f_0$, multiplying ψ_a from the right, gives

$$\begin{aligned}\psi_a\epsilon_f{}^0 f_0 &= [G_{1+}(e_0 + e_3) + G_{2-}(e_1 - ie_3) + G_{2+}(f_1 - if_2) + G_{1-}(f_0 + f_3)](\epsilon_f{}^0 f_0)\\ &= [G_{1+}^*(f_0 - f_3) - G_{2-}^*(f_1 + if_2) - G_{2+}^*(e_1 + ie_2) + G_{1-}^*(e_0 - e_3)]\epsilon_f{}^0 e_0\end{aligned} \tag{6.18}$$

Thus, an f_0 charge coupling on the right, doubles the internal degrees of freedom of ψ_a from 4 to 8, just as quaternion charge $\epsilon^k e_k$ did in the old matrix formalism (Edmonds, 1973). Here, however, we have three possible new 'charge' couplings, $\epsilon_e{}^k$, $\epsilon_f{}^0$, and $\epsilon_f{}^k$, instead of just one, $\epsilon_e{}^k$.

We note a special feature of f_0. It can be used to introduce a ψ_v term into the ψ_a equation

$$\begin{aligned}A(\epsilon_f{}^0 f_0)\psi_v \to A'(\epsilon_f{}^0 f_0)\psi_v' &= \mathscr{L}^* A \mathscr{L}(\epsilon_f{}^0 f_0)\mathscr{L}^*\psi_v\\ &= \mathscr{L}^* A \mathscr{L}\mathscr{L}^\ddagger(\epsilon_f{}^0 f_0)\psi_v = \mathscr{L}^*(A(\epsilon_f{}^0 f_0)\psi_v)\end{aligned} \tag{6.19}$$

This reminds us of the m coupling for the Weyl equations for ψ_a and ψ_v, so we could consider

$$\begin{pmatrix}(P_e + P_f) & 0\\ 0 & (P_e + P_f)^{\ddagger *}\end{pmatrix}\begin{pmatrix}\psi_a\\ \psi_v\end{pmatrix} = \begin{pmatrix}0 & m^{\ddagger *}\\ m & 0\end{pmatrix}\begin{pmatrix}\psi_a\\ \psi_v\end{pmatrix}$$

However, this does not seem to generate any really 'new' structure.

The generalisation of P to $P_e{}^\mu e_\mu + P_f{}^\mu f_\mu$, raises questions of finding a conserved current J to act as a source of A. Indeed, it brings into question the meaning of conserved itself. The simplest covariant generalisation would be to require that $(P_e + P_f|J_e + J_f) = 0$, where $(P|J)$ is given by equation (5.6). This inner product is, in the most general case, not invariant. The usual requirement of assuming a conserved current is perhaps an oversimplification of nature, in the fully relativistic domain where virtual transmutation is common and the concept of a single type field in space an obvious oversimplification.

Below we consider the usual Dirac current conservation in this extended quaternion algebra, for the case $P_f{}^k\psi \equiv 0$ and $P_f{}^0\psi \equiv -m\psi$:

$$\begin{aligned}(P_e{}^\mu e_\mu + P_f{}^0 f_0|J^\nu e_\nu) &\equiv \tfrac{1}{2}[P^\vee J + (P^\vee J)^\wedge]\\ &= (P_e{}^\mu J_\mu)e_0 - [P_f{}^{0*} f_0 J^\mu e_\mu] + [P_f{}^{0*} f_0 J^\mu e_\mu]^*\end{aligned} \tag{6.20}$$

But $P_f{}^{0*} \equiv P_f{}^0$ and $J_\mu{}^* \equiv J_\mu$, therefore

$$(P|J) = (P_e{}^\mu J_\mu)e_0 - (P_f{}^0 J_e{}^\mu)f_\mu \tag{6.21}$$

This is not invariant. As we shall see, $(P_f{}^0 J_e{}^\mu) = -mJ_e{}^\mu$, so the f_μ term will not go away. We can overcome this difficulty by again redefining the covariant inner product for 8-vectors.

We can still define

$$(P|J)_{\text{II}} \equiv \tfrac{1}{2}[(P^{\vee}J) + (P^{\vee}J)^{\vee}] \tag{6.22}$$

provided care is taken in evaluating the second term. Recall that $(e+f)^{\vee} = e^{\ddagger} - f^{*}$. Expanding P and J, we find

$$\begin{aligned} 2(P|J)_{\text{II}} &= [(P_e{}^{\ddagger} - P_f{}^{*})(J_e + J_f)] + [\ \ ]^{\vee} \\ &= [P_e{}^{\ddagger}J_e - P_f{}^{*}J_e + P_e{}^{\ddagger}J_f - P_f{}^{*}J_f] + [\ \ ]^{\vee} \\ &= (P_e{}^{\ddagger}J_e - P_f{}^{*}J_f) + (\ \ )^{\ddagger} + (P_e{}^{\ddagger}J_f - P_f{}^{*}J_e) - (\ \ )^{*} \end{aligned} \tag{6.23}$$

Under a Lorentz transformation this becomes

$$2(P'|J')_{\text{II}} = [(P_e{}^{\ddagger}J_e - P_f{}^{*}J_f) + (\ \ )^{\ddagger}] + \mathscr{L}^{\ddagger}[(P_e{}^{\ddagger}J_f - P_f{}^{*}J_e) - (\ \ )^{*}]\,\mathscr{L} \tag{6.24}$$

It would appear to not be invariant, in general. Notice that

$$P_e{}^{\ddagger}J_f - P_f{}^{*}J_e \rightarrow P_e^{\mu *}J_f{}^{\nu}\sigma_\mu\sigma_\nu - P_f^{\nu *}J_e{}^{\mu}\sigma_\nu\sigma_\mu \tag{6.25}$$

Obviously this is imaginary and not equal to zero for $\mu \neq 0$, $\nu \neq 0$, and $\mu \neq \nu$. Therefore $(P|P)_{\text{II}} \propto e_0$ only if $P_f{}^k = 0$. Also, $(P|J)_{\text{II}} \propto e_0$ if $J_f{}^k = 0$ and $P_f{}^kJ = 0$. Therefore, for this inner product, we can write

$$(P_e{}^{\mu}e_\mu + P_f{}^0 f_0 | J_e{}^{\mu}e_\mu)_{\text{II}} = (P_e{}^{\mu}J_{e\mu})e_0 + Of_\mu \tag{6.26}$$

Both of these inner products transform the same way and either can be used in the generalised Maxwell equation. This latter form seems most 'compatible' with 5-vectors and the former form with 8-vectors, should any physically exist.

We can now examine the conservation law for the current 4-vector, using equation (6.26). For the 5-vector wave equation

$$(P_e{}^{\mu}e_\mu + P_f{}^0 f_0)\psi_a \equiv (P_e{}^{\mu}e_\mu + -mf_0)\psi_a = 0 \tag{6.27}$$

we find, concentrating on only the first term in j^{μ},

$$\begin{aligned} P_e{}^{\mu}(\psi_a{}^{\dagger}e_\mu\psi_a) &= (P_e{}^{\mu}\psi_a{}^{\dagger})e_\mu\psi_a + \psi_a{}^{\dagger}e_\mu(P_e{}^{\mu}\psi_a) \\ &= (P_e{}^{\mu *}\psi_a)^{\dagger}e_\mu\psi_a + \psi_a{}^{\dagger}(e_\mu P_e{}^{\mu}\psi_a) \\ &= (e_\mu{}^{\dagger}(P_e{}^{\mu *}\psi_a))^{\dagger}\psi_a + \psi_a{}^{\dagger}mf_0\psi_a \end{aligned} \tag{6.28}$$

Since $e_\mu{}^{\dagger} = e_\mu{}^{*} = e_\mu$ and earlier $(e_\mu P_e{}^{\mu})^{*} \equiv e_\mu P_e{}^{\mu}$ was found necessary, we get $(e_\mu P_e{}^{\mu}\psi_a)^{\dagger}\psi_a$ for the first term in equation (6.28). As a result, equation (6.28) does not give zero. In our previous work we got zero here by separating $i\hbar\partial^{\mu}$ and putting the $i\hbar$ term with m. It was then assumed that $\partial^{\mu *} = \partial^{\mu}$ and $(\partial|J)$ was computed. This seems inconsistent and trumped up. This discussion may be showing that the usual conserved current ideas are naive, or simply that the extended quaternion formalism is awkward in dealing with conserved currents. In curved space-time and for the 8-vector wave equation momentum operator, the conserved current idea certainly becomes complicated. After all, those present problems with ordinary relativistic quantum theory, involving negative probabilities, are related to the probability interpretations of the current 4-vector. The space-time extensions considered here, indicate that probability conservation is much more subtle in the 5- or 8-dimensional space-time.

7. *Curved Space Considerations*

The 5- and 8-vector ideas introduced here, can be easily extended to curved space-time, using the ideas we recently put forward for the quaternion formulation (Edmonds, 1974). We restrict our attention to the Dirac equation for brevity, since the rest is straightforward.

In curved space we have

$$\begin{aligned} P_e{}^\mu e_\mu + P_f{}^\mu f_\mu &\to P_e{}^\mu b_\mu(x) + P_f{}^\mu h_\mu(x) \\ b_\mu(x) = b_\mu{}^{(\alpha)}(x) e_\alpha, &\qquad h_\mu(x) = h_\mu{}^{(\alpha)}(x) f_\alpha \end{aligned} \tag{7.1}$$

To define the space-time dependent quaternion elements, we use

$$b_\mu \equiv L^*(x) e_\mu L(x) \qquad h_\mu \equiv L^*(x) f_\mu L(x) \tag{7.2}$$

and

$$b_\mu{}^\ddagger (D^\mu D^\nu - D^\nu D^\mu) b_\nu = b_\mu{}^\ddagger T^{\mu\nu} b_\nu \tag{7.3}$$

Equation (7.3) can be used to find $b_\mu(x)$, or equivalently $L^\mu(x) e_\mu$. Notice that, since L is assumed to be an L_e like $\mathscr{L}$ was,

$$b_\mu{}^* = (L^* e_\mu L)^* = b_\mu \qquad h_\mu{}^* = (L^* f_\mu L) = L^\ddagger f_\mu{}^* L^{\ddagger *} \tag{7.4}$$

so that

$$h_0 = L^* f_0 L = f_0 L^\ddagger L \qquad \text{and} \qquad h_0{}^* = f_0 L^* L^{\ddagger *} = f_0 (L^\ddagger L)^* \tag{7.5}$$

Thus, h_0 is not real unless $L^\ddagger L$ is real. In general $L^\ddagger L \neq 1 e_0$, but is, instead, some function of x.

The covariant derivative in equation (7.3) is defined by requiring that

$$D^\mu (g_{\nu\lambda}) \equiv 0 \qquad g_{\mu\nu} \equiv \tfrac{1}{2}[(b_\mu{}^\ddagger b_\nu) + (\ )^\ddagger] \propto e_0 \tag{7.6}$$

But for the generalised Dirac equation, we need

$$(i\hbar b_\mu D_e{}^\mu + ik h_\mu D_f{}^\mu) \psi_a = 0 \tag{7.7}$$

and $D_f{}^\mu$ remains undefined. We could define it by extending equation (7.6):

$$D_f{}^\mu (g_{(f)\nu\lambda}) \equiv 0 \qquad g_{(f)\mu\nu} \equiv \tfrac{1}{2}[(h_\mu{}^\vee h_\nu) + (\ )^\vee] \propto e_0 \tag{7.8}$$

This extension includes equation (7.6)—just replace (f) by (e).

As in the quaternion formalism, we can define $\psi_a = \psi_a{}^\mu \ell_\mu(x)$ where

$$\ell_{(e)\mu}(x) \equiv L^\ddagger e_\mu \qquad \ell_{(f)\mu}(x) \equiv L^\ddagger f_\mu \tag{7.9}$$

Then

$$D^\nu \ell_{(e)\mu}(x) = D^\nu (L^\ddagger e_\mu) = D^\nu (L^\ddagger) e_\mu \tag{7.10}$$

The covariant derivative of L comes from

$$D^{\mu}(g_{\nu\lambda}) \equiv 0 \qquad \text{and} \qquad b_{\mu} \equiv L^{*}e_{\mu}L \tag{7.11}$$

To evaluate $D^{\mu}(\psi_{a}{}^{\nu})$, we use equation (7.10) and the requirement that equation (7.7) must be form invariant under the Lorentz Principle.

This is only an outline of the curved space formalism. We include it to show that the 5- and 8-dimensional space-time extensions can be readily generalised to include curved space. Thus quantum field equations and gravitation can be blended in the enlarged space in a straightforward way. Quantisation of these equations is not straightforward and poses formidable difficulties.

The curved space generalisation shows how t and T, in the 5-vector $(dtb_0 + dx^k b_k + dTh_0)$, are algebraically distinct, even though in the preferred frame $dt = dT$. The space-time dependence of b_0 and h_0 is quite different:

$$b_0(x) = L^{*}e_0 L = e_0(L^{*}L), \qquad h_0(x) = L^{*}f_0 L = f_0(L^{\ddagger}L) \tag{7.12}$$

Ultimately, we must consider the swarm of partons as a master field, with some kind of wave equation or coupled equations.

Particular composite states of this swarm are then 'stable' eigenstates of the cosmic time operator $ik\partial/\partial T$. This would be analogous to the stable swarms of protons, neutrons, and electrons we call atoms which have definite masses (and some of which are unstable). These composite states have internal and translational energy states, related to the local (frame dependent) time operator $i\hbar\partial/\partial t$. The parton dynamics themselves are presently unknown. They may involve extended quaternion coupling constants or be otherwise radically different. Hopefully, the cosmic time operator will open productive new approaches to the mass spectrum problem.

8. *Algebraic Isomorphism*

We have seen that the complex quaternion algebra naturally describes space-time and particle dynamics, in the absence of rest mass. We have also seen that rest mass leads to a generalisation from complex quaternions to a (16 real number) algebraic number system. This generalisation involves extending the complex number system $\{e_0, ie_0\}$ to a noncommutative ring $\{e_0, ie_0, f_0, if_0\}$. This extension, from the way we introduced it, appears somewhat artificial and 'cooked up' to stretch the quaternion number system in a semiarbitrary way. After completing the development as described above, I found that it is not arbitrary at all.

The extension discussed is *the* natural extension of complex quaternions, as we shall now show. (That it so nicely accommodates rest mass, is probably no accident!)

Hamilton originally invented the quaternion algebra in 1843 to generalise the vector ideas of complex variables, $P_x + iP_y$, to three dimensions. His 'regular' quaternions are four part numbers $a = a^0 + a^k i\sigma_k$ where $P_x i\sigma_1 + P_y i\sigma_2 + P_z i\sigma_3$ gives the desired generalisation. (It was not appreciated until long after Minkowski space, that $P_t 1$ gives the fourth part of the quaternion, meaning that the quaternion formalism is really tailored to relativistic physics.) The basis elements here are $\{1, i\sigma_k\}$. They form a group under multiplication. It turned out that complex quaternions, obtained by letting a^0 and a^k become complex, were required to describe massless relativistic quantum theory. This larger algebra has *another* closed subalgebra, in addition to $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$. This other subgroup is $\{1, i\sigma_1, \sigma_2, \sigma_3\}$. Unlike the regular quaternions, it forms a noncommutative ring (instead of a noncommutative field). Notice that

$$(a^0 - a^k i\sigma_k)(a^0 + a^k i\sigma_k) = (a^0)^2 + (a^k a^k) \geqslant 0$$

$$(a^0 - a^1 i\sigma_1 - a^2\sigma_2 - a^3\sigma_3)(a^0 + a^1 i\sigma_1 + a^2\sigma_2 + a^3\sigma_3) = [(a^0)^2 + (a^1)^2] - [(a^2)^2 + (a^3)^2] \tag{8.1}$$

Each of these two subgroups has the further subgroup $\{1, i\sigma_1\}$ which is isomorphic to $\{1, i\}$, the complex numbers. The regular quaternion algebra can thus be generalised, symbolically, by forming $\{1, i\sigma_1\} \otimes \{1, i\sigma_k\}$, $k = 1, 2, 3$. This is the algebra of massless relativistic physics. What then do we get when we form the next larger natural algebra? You guessed it. The algebra $\{1, i\sigma_1, \sigma_2, \sigma_3\} \otimes \{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ is isomorphic to $\{e_0, ie_0, e_k, ie_k, f_0, if_0, f_k, if_k\}$. By comparing the multiplication tables, we find that

$$1 \otimes 1 = e_0 \qquad 1 \otimes i\sigma_k = ie_k \qquad i\sigma_1 \otimes 1 = ie_0$$
$$i\sigma_1 \otimes i\sigma_k = -e_k \qquad \sigma_2 \otimes 1 = f_0 \qquad \sigma_2 \otimes i\sigma_k = if_k \tag{8.2}$$
$$\sigma_3 \otimes 1 = if_0 \qquad \text{and} \qquad \sigma_3 \otimes i\sigma_k = -f_k$$

Therefore, $\{1, i\sigma_1, \sigma_2, \sigma_3\}$ is isomorphic to $\{e_0, ie_0, f_0, if_0\}$.

The wavefunctions can be considered as functions of the ring (e_0, ie_0, f_0, if_0), multiplying e_μ from the left. For particles without mass, presumably the wavefunctions consist of functions of the field (e_0, ie_0), multiplying e_μ from the left (at least when no interactions with massive particles are considered). This restriction appears to also be true of massive particles, in the nonrelativistic limit—the Pauli equation. This is probably why the generalisation had been overlooked until now. It would appear that the analytic function machinery, used in nonrelativistic quantum theory (dispersion relations, complex angular momentum, poles, etc.), should be generalised to a 4-dimensional hypercomplex plane for relativistic quantum theory.

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References

A fairly complete listing of the background literature is to be found in our previous quaternion papers: *International Journal of Theoretical Physics*, Vol. 6, No. 3, p. 205 (1972), Vol. 7, No. 6, p. 475 (1973); *Foundations of Physics*, **3**, 313 (1973), to be published (1974); *Letters Nuovo Cimento*, **5**, 572 (1972), 7, 398 (1973), to be published. See especially *American Journal of Physics*, **42**, 220 (1974), for early quaternion references.